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Photon splitting in strong magnetic fields: asymptotic approximation formulae vs. accurate numerical results

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Abstract

We present the results of a numerical calculation of the photon splitting rate below the electron-pair creation threshold ($\omega \leq 2m$) in magnetic fields $B \gtrsim B_{\rm cr} = m^2/e = 4.414 \times 10^9$ T. Our results confirm asymptotic approximations derived in the low-field ($B < B_{\rm cr}$) and high-field ($B \gg B_{\rm cr}$) limit, and allow interpolating between the two asymptotic regions. Our expression for the photon splitting rate is a simplified version of a formula given by Mentzel et al. We also point out that, although the analytical formula is correct, the splitting rates calculated there are wrong due to an error in the numerical calculations.

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The exotic process of magnetic photon splitting, i.e. the decay of a photon into two photons in the presence of a very strong magnetic field, has recently attracted renewed attention, mainly because of the great importance this process may have in the interpretation of the spectra of cosmic γ-ray burst sources. The basic formulae for magnetic photon splitting had already been derived in the seventies [1–5]. In the first approach the photon splitting effect was analyzed using the Heisenberg-Euler effective Lagrangian. This method is justified under the condition $\omega \ll m$. The result was the $(\omega/m)^5(B/B_{\rm cr})^6$ dependency of the attenuation coefficient in the weak field regime ($B \ll B_{\rm cr}$) [1,2]. Adler [3] was the first to solve the problem for arbitrary magnetic field strengths and photon energies below the pair creation threshold. He used the gauge invariant proper-time method and presented numerical results up to $B = B_{\rm cr}$ for the two cases $\omega = m$ and $\omega \ll m$. Other gauge-invariant versions of the splitting-amplitude were found later by Stoneham [5], Baier et al. [6], and recently, using a path-integral approach, by Adler and Schubert [7]. Nevertheless, numerical results, apart from those obtained by Adler, and in particular for magnetic fields exceeding $B_{\rm cr}$, were not available. Mentzel et al. [8] therefore undertook a rederivation of the photon splitting rate using a configuration space representation for the electron propagator in a strong magnetic field with the aim to find an exact analytical expression that would be suitable for numerical evaluation for magnetic fields above $B_{\rm cr}$. The representation chosen for the electron propagator, though not gauge-invariant, had the advantage that the final expression for the photon splitting rate did not contain any singularities for photon energies below $\omega = 2m$. Moreover, the sums over the Landau-states that had to be calculated in this method were expected to converge very rapidly for $B > B_{\rm cr}$.

The numerical results presented in [8] seemed to suggest that the absolute values of the rates were much bigger than was previously assumed, but failed to reproduce the asymptotic low-frequency, low-field behavior that followed from the earlier investigatons. Consequently, these results were criticized by Adler [9], who suggested that the fixing of the gauge had led to calculation errors that could not be detected in the final expression.

More recently, Baier et al. [10] determined the behavior of the photon splitting rate in the limit of extremely high magnetic fields, $B/B_{\rm cr} \gg 1$, and demonstrated that in this case the splitting rate becomes independent of field strength. In agreement with these results are the recent results of Heyl and Hernquist [11] who reexamined the effective Lagrangian method and were able to calculate the attenuation coefficient for arbitrary magnetic field strengths and $\omega \ll m$.

In this note we will reexamine the formula given by Mentzel et al. [8], simplify it, evaluate it numerically, and compare with the results of the asymptotic formulae obtained by Adler [3], Baier et al. [10], and Heyl and Hernquist [11]. This will allow to quantitatively assess the range of validity of the different asymptotic results. We point out that the numerical results of [8] are in error because of an incorrect sign in the code for evaluating the analytical formula for magnetic photon splitting.

Following the paper by Baier et al. [10] we shall concentrate on the process $\perp \rightarrow \parallel \parallel$, where a photon with the polarization vector parallel to the plane formed by the $\vec{k}$ vector and the magnetic field axis decays into two photons with polarization vectors perpendicular to that plane. This is expected to be the dominating decay channel as soon as dispersive effects become important. Note that these designations for the polarizations are exactly opposite to those chosen by Adler. Furthermore, we confine ourselves to photon energies below the

pair creation threshold $\omega < 2m$, above which pair creation is expected to dominate photon splitting.

The complexity of the general expressions for the photon splitting rate in magnetic fields has always been an impediment to quantitative studies of the importance of the photon splitting process in actual astrophysical model calculations. The availability of appropriate approximation formulae, and a knowledge of their respective ranges of applicability, is therefore a prerequisite for such investigations [12]. On the low-frequency, weak-field ($B \lesssim B_{\rm cr}$) side, the attenuation coefficient can be written (in the notation of Stoneham [5])

$$\ell^{-1}(\perp \rightarrow \parallel \parallel) = \frac{\alpha^3 m}{60\pi^2} \left(\frac{B}{B_{\rm cr}}\right)^6 \left(\frac{\omega}{m}\right)^5 \times (M_1(B))^2 \ , \tag{1}$$

where the scattering amplitude $(M_1(B))$ is given by eq. 41 of ref. [5]. Adler [3] demonstrated that for field strenghts up to $\sim 0.1\ B_{\rm cr}$ the coefficient is excellently reproduced by that equation, with M_1 replaced with its $B = 0$ value, $M_1(0) = 26/315$.

On the very high field side, $B \gg B_{\rm cr}$ Baier et al. [10] have recently shown that, for given energy below $2m$, the attenuation coefficent tends towards a constant, magnetic field independent value, viz.

$$\ell^{-1}(\perp \rightarrow \parallel \parallel) = \int \frac{1}{32\pi} |T_{B \gg B_{\rm cr}}|^2 \frac{\mathrm{d}\omega'}{\omega^2}, \tag{2}$$

where $T_{B \gg B_{\rm cr}}$ is the high-field approximation of the splitting amplitude:

$$T_{B \gg B_{\rm cr}} = \frac{2(4\pi\alpha)^{3/2}}{\pi^2} \Big[\frac{\omega' m^2}{\omega'' \sqrt{4m^2 - \omega''^2}} \arctan \left(\frac{\omega''}{\sqrt{4m^2 - \omega''^2}} \right) + \frac{\omega'' m^2}{\omega' \sqrt{4m^2 - \omega'^2}} \arctan \left(\frac{\omega'}{\sqrt{4m^2 - \omega'^2}} \right) - \frac{1}{4}\omega \Big] \ . \tag{3}$$

On the basis of an accurate numerical evaluation of the general expression for the photon splitting rate we will now determine the range of field strengths and frequencies from where on the high-field asymptotic behavior is valid, and provide data for the range of intermediate field strengths.

The general expression we evaluate is the one derived in [8]. Two important remarks on this work are in order. The first is that by reason of a simple error in sign in the computer code (the quantity K_2 defined in eq. (A12) in the Appendix of ref. [8] contained a plus sign in the code instead of a minus sign in the first term on the right-hand side) all the *numerical* results are wrong. This error escaped the numerical checks of the polarization selection rules. Speculations that gauge invariance problems caused the deviations from previous results are thus found to be insubstantial. The second remark is that the *analytical* expression obtained by Mentzel et al. [8] is *correct*. The low-frequency, weak-field limit (1) can be derived from it analytically, and the results of the numerical evaluation (corrected for the error in sign) are in agreement with previous results in all ranges of parameters where a comparison is possible. We note that by a suitable rearrangement of terms the numerical evaluation of the expression of [8] could be speeded up by two orders of magnitudes. The main step in these optimizations is the summing up of the different polarization states. We performed this sum using the computer algebra program Maple. Once the summation has been carried out, the

remaining integration can be reduced to 5 integrals that can be evaluated independendly from the rest of the formula.

An important point with regard to the accuracy of the results is the inclusion of sufficiently many Landau states in carrying out the sums over the intermediate states. In particular for magnetic fields in excess of $B_{\rm cr}$ these sums are found to converge very rapidly.

In Fig. 1 we present, for the three frequencies $\omega/m = 0.1,\ 1.0,\ 1.99$, the results of the numerical evaluation (crosses) of the general expression for the magnetic photon splitting attenuation coefficient for the decay channel $\perp \rightarrow \parallel\ \parallel$ as a function of the field strength in comparison with the predictions of the asymptotic formulae. The steep solid line on the left corresponds to the $\omega^5\, B^6$ dependence of the low-field formula eq. (1) (with $M_1(0)$ inserted for $M_1(B)$), while the dotted horizontal line corresponds to the high-field limit (2). The figure shows how, for all three frequencies, the numerical results approach the low-field limit as B decreases, and that the numerical values tend towards the high-field limit as B increases. The results are in good agreement with those of Heyl and Hernquist [11]. We find that their formulae have a very high accuracy for photon energies up to $\omega = 0.5m$. The high-field limit of Baier et al. [6] is reached, to within a few per cent, already for magnetic field strengths $B/B_{\rm cr}$ around 30, independent of the photon-energy.

We now proceed to results for the differential splitting rate. This quantity provides the information on the probability with which a certain partitioning of the energy of the decaying photon on the outgoing photons is assumed. Therefore it is reasonable to normalize this quantity to unity. Note that in this respect our procedure differs from that of Baier et al. [10], who used a magnetic-field dependent normalization constant, which blurs the comparison of the results at different field strengths.

In Fig. 2, for the energies $\omega/m = 0.1$, $\omega/m = 1.5$ and $\omega/m = 1.99$ and magnetic field strengths $B = B_{\rm cr}$, $5\ B_{\rm cr}$ and $100\ B_{\rm cr}$ numerical results are shown for the differential splitting rate (normalized to unity when integrated over the abscissa) as a function of ω'/ω. For $B = 100\ B_{\rm cr}$, as expected, the shape of the differential splitting rate is exactly identical, for all three photon energies, to the one determined from Baier's asymptotic expression (solid curves in Fig. 2). This agreement, however, is seen to be present even for the smaller magnetic field strengths of $B = 5\ B_{\rm cr}$ and $B = B_{\rm cr}$, and it is only for photon energies approaching the pair creation threshold in the latter case that sizeable deviations from the asymptotic behavior appear. We also see from Fig. 2 that the plateau-forming effect described by Baier et al. [10] occurs already below $B = 5\ B_{\rm cr}$. Thus we have the result that the high-field asymptotic formulae represent a good approximation even at field strengths where they were not a priori expected to apply. This opens the possibility of carrying out astrophysical model calculations using the asymptotic high-field formulae for $B \gtrsim 5B_{\rm cr}$, field strengths that are supposed to be present in soft γ repeaters [12,13].

In this note we have presented numerically accurate values of attenuation coefficients for photon splitting in strong magnetic fields, and have used these results to determine the ranges of parameters in which asymptotic approximation formulae can be applied in the place of the full complicated expression for the photon splitting rate. The results should stimulate, in high-energy astrophysics, new quantitative studies of the role of the exotic process of photon splitting in the formation of high-energy spectra of strongly magnetized cosmic objects.

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FIGURES

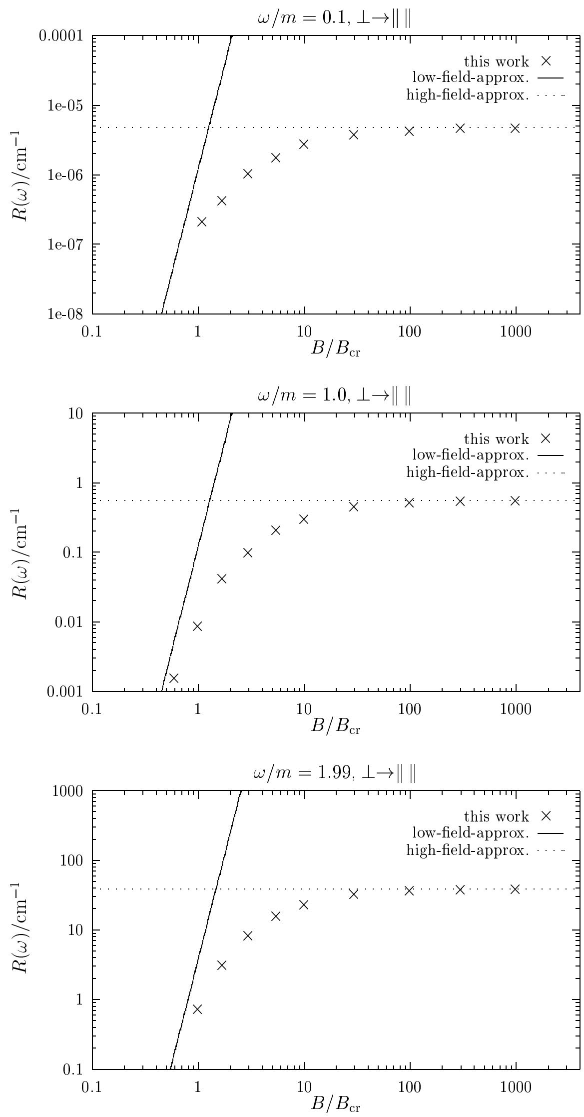

FIG. 1. Total attenuation coefficient for the splitting channel $\perp\to\parallel\,\parallel$ as a function of field strength for the three photon frequencies $\omega/m = 0.1$, 1.0, and 1.99. Crosses denote the results of the present calculations, the weak-field result is represented by the solid straight line, the high-field approximation by the horizontal dotted line.

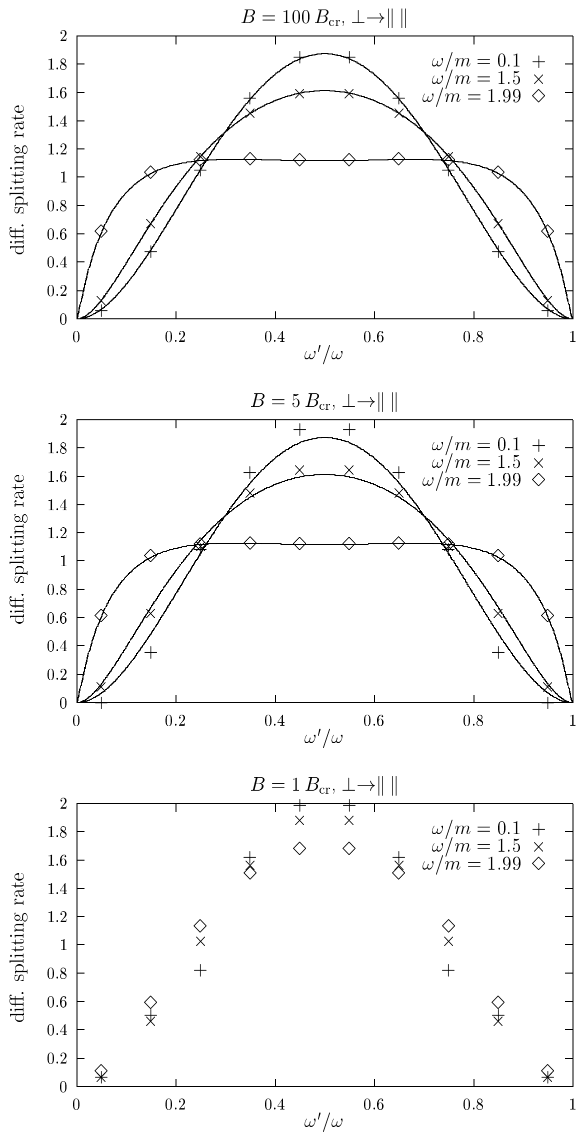

FIG. 2. Differential splitting rate (normalized to unity when integrated over the abscissa) for the splitting channel $\perp\to\parallel\,\parallel$ and three frequencies ω of the incoming photon as a function of the ratio of the frequency ω' of one of the outgoing photons and ω, for the magnetic field strengths (from top to bottom) $B/B_{\rm cr} = 100$, 5, and 1. The solid curves represent the results of the asymptotic high-field approximation by Baier et al. This approximation is seen to be valid even at magnetic field strengths as low as $B \sim 5B_{\rm cr}$.